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THE EFFECTS OF DISPERSION AND DISSIPATION ON WAVE PROPAGATION IN VISCOELASTIC LAYERED COMPOSITIES

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Abstract—Transient waves propagating normal to the layerings of a semi-infinite viscoelastic layered composite are studied. The stress response at a large distance from the boundary due to a unit step stress applied at the boundary is obtained. If the distance is not large enough, the stress response is oscillatory due to the dispersive nature of the composite. As the distance increases, the dissipation effect of the viscoelastic materials becomes pronounced and eventually wipes out completely the oscillatory response. The transition from the oscillatory response to the monotonic response is controlled by a parameter γ which contains (a) the impedance mismatch of the composite which contributes to the dispersion, (b) the dissipative properties of the viscoelastic materials and (c) the distance traveled by the wave.

1. INTRODUCTION

Consider a semi-infinite periodic layered composite as shown in Fig. 1 in which each period 2ω consists of two layers of homogeneous, isotropic, linear viscoelastic materials. The thicknesses of individual layers are $2h_i$ $(i = 1, 2)$ where the subscripts 1 and 2 refer to materials 1 and 2, respectively. We will consider plane wave propagation in the direction x which is normal to the layers. For the problem considered here, the surface $x = 0$ need not be the central surface of the first layer. We will assume, however, that the first layer in which $x = 0$ is located is occupied by material 1.

The composite is initially at rest and at time $t = 0$, time-dependent, uniformly-distributed normal and shear stresses are applied at the surface $x = 0$. Since the problem considered is linear, the solutions due to the applied normal stress and the shear stress can be treated separately. The two solutions are mathematically identical. Therefore, we will consider only the solution due to the applied normal stress at $x = 0$. Moreover, we will assume that the applied normal stress at $x = 0$ is the Heaviside unit step function in time t, because the solution for a more general applied normal stress can be obtained by a linear superposition.

The stress response at a position x which is sufficiently large can be obtained by an asymptotic analysis. When both layers are elastic, the solution can be expressed in terms of an integral of an Airy function [1, 2]. The stress, as a function of time t, oscillates around the Heaviside step function. When one or both layers are viscoelastic, the asymptotic solution can be expressed in terms of an error function [1, 2]. The stress response is no longer an oscillatory function of t, but a monotonically increasing function of t which approaches to the unit stress as t increases.

Since elastic materials are special cases of viscoelastic materials, one might ask how a monotonic solution becomes an oscillatory solution when the viscoelastic materials become elastic. Alternately, one might ask what would be the behavior of the asymptotic solution if the relaxation functions of the viscoelastic materials are nearly step functions. Clearly, when the position x is not large enough, the dissipative effect of the viscoelastic materials does not have enough time to prevail and the stress response is essentially governed by the dispersive nature of the composite which causes the solution to be oscillatory. As x increases, the dissipative effect, no matter how small, becomes prominent and dampens the despersive mechanism so that the solution is non-oscillatory. The purpose of this paper is to study the effects of the dispersion, dissipation and the distance of wave propagation have on the wave profile. To simplify the analysis, we will consider only solutions at $x = 2\omega N$ where N is an arbitrary positive integer.

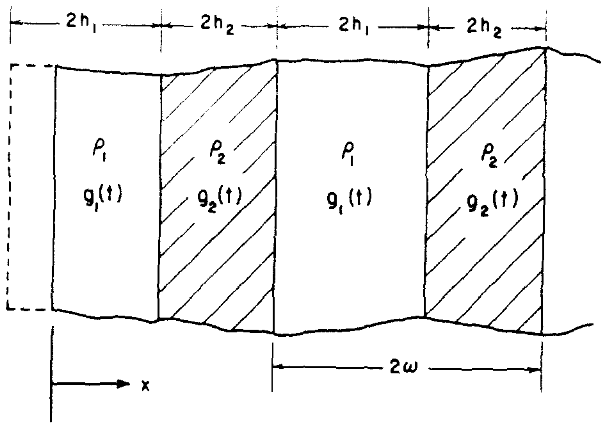

Fig. 1. Geometry of the viscoelastic layered composite.

It should be pointed out that a similar problem was studied by Sve [1] for two special viscoelastic materials. The imaginary part of the wave number for the special materials is assumed to be proportional to the absolute value of the frequency or proportional to the square of the frequency. Hegemier[3] obtained asymptotic solutions for elastic composites as well as viscoelastic composites. However, his solutions differ from that obtained here and in [2]. A discussion on the differences will be given later.

2. SOLUTION FOR $x = 2\omega N$

The equations of motion and the continuity of the displacement are given by

$$\frac{\partial \sigma_i}{\partial x} = \rho_i \dot{v}_i, \qquad (i = 1, 2) \qquad (1)$$

$$\frac{\partial v_i}{\partial x} = \dot{\varepsilon}_i, \qquad (i = 1, 2) \qquad (2)$$

where σ_i, ε_i, v_i, ρ_i $(i = 1, 2)$ are the normal stress, normal strain, normal particle velocity and mass density, respectivly. A dot stands for differentiation with respect to time t. The initial and boundary conditions are

$$\sigma_i(x, 0^-) = v_i(x, 0^-) = \varepsilon_i(x, 0^-) = 0 \qquad (i = 1, 2) \qquad (3)$$

$$\sigma_i(\infty, t) = 0 \qquad (i = 1, 2) \qquad (4)$$

$$\sigma_1(0, t) = H(t) \qquad (5)$$

where $H(t)$ is the Heaviside unit step function. The relation between σ_i and ε_i is written in the form of Stieltjes integral

$$\sigma_i(x, t) = \int_{0^-}^{t} g_i(t - t')\, d\varepsilon_i(t') \qquad (i = 1, 2) \qquad (6)$$

where $g_i(t)$ are the relaxation functions of the viscoelastic layers.

Let $\bar{f}(p)$ be the Laplace transform of $f(t)$:

$$\bar{f}(p)=\int_{0^-}^{\infty} f(t)\,e^{-pt}\,dt. \tag{7}$$

Equations (1)–(6) then reduce to

$$\frac{\partial^2\bar{\sigma}_i}{\partial x^2}=k_i^2\bar{\sigma}_i \tag{8}$$

$$\bar{\sigma}_i(\infty,p)=0,\qquad \bar{\sigma}_1(0,p)=\frac{1}{p} \tag{9}$$

where

$$k_i=\sqrt{\rho_i p/\bar{g}_i}. \tag{10}$$

Since k_i is periodic in x with periodicity 2ω, by using the Floquet theory [4] the solution for $x=2\omega N$ where N is an arbitrary positive integer is

$$\bar{\sigma}_1(x,p)=\frac{1}{p}\,e^{-\kappa x} \tag{11}$$

where κ is the characteristic exponent given by (see [2, 5, 6])

$$\cosh(2\omega\kappa)=\theta\cosh(2k_1h_1+2k_2h_2)-(\theta-1)\cosh(2k_1h_1-2k_2h_2) \tag{12}$$

$$\theta=\tfrac{1}{4}\left(\frac{\rho_1k_2}{\rho_2k_1}+\frac{\rho_2k_1}{\rho_1k_2}+2\right). \tag{13}$$

Therefore, the solution for $x=2\omega N$ is

$$\sigma_1(x,t)=\frac{1}{2\pi i}\int_{Br}\frac{1}{p}\,e^{pt-\kappa x}\,dp. \tag{14}$$

For a large x, the main contribution to the Bromwich contour integral of eqn (14) appears to come from the values of integrand near $p=0$. Hence we must study the behavior of κ near $p=0$ before we evaluate eqn (14) for large x.

3. BEHAVIOR OF κ NEAR $p=0$

For most viscoelastic materials, the relaxation function $g_i(t)$, $(i=1,2)$ is a monotonically decreasing function of t. Let $g_{i\infty}$ be the value of $g_i(t)$ at $t=\infty$. For most viscoelastic solids $g_{i\infty}$ is non-zero. If $\bar{g}(p)$ is the Laplace transform of $g_i(t)$,

$$\begin{aligned} p\bar{g}_i(p)&=p\int_0^{\infty} g_i(t)\,e^{-pt}\,dt\\ &=g_{i\infty}+p\int_0^{\infty}[g_i(t)-g_{i\infty}]\,e^{-pt}\,dt. \end{aligned} \tag{15}$$

For small $p, e^{-pt} = 1 - pt + \ldots$ Hence

$$p\bar{g}_i(p) = g_{i\infty}(1 + a_i p - a_i T_i p^2 + \ldots) \tag{16}$$

where

$$\left.\begin{aligned} a_i g_{i\infty} &= \int_0^\infty [g_i(t) - g_{i\infty}]\, dt \\ T_i &= \frac{1}{a_i g_{i\infty}} \int_0^\infty [g_i(t) - g_{i\infty}] t\, dt \end{aligned}\right\} . \tag{17}$$

It is seen that $a_i g_{i\infty}$ is the area between the curve $g_i(t)$ and the horizontal line $g_i(t) = g_{i\infty}$ while T_i is the distance of the centroid of this area from $t = 0$. According to [7], a_i provides a measure of the "viscosity" of the viscoelastic materials. An example of relaxation function which yields eqn (16) is the standard linear viscoelastic solid

$$g_i(t) = g_{i\infty}\left(1 + \frac{a_i}{T_i} e^{-t/T_i}\right). \tag{18}$$

Using eqns. (16), (10) and (13), the right-hand side of eqn (12) can be expanded into power series in p. If we assume that, for small p, κ can be expressed as

$$c_\infty \kappa = p - \frac{\upsilon}{2!} p^2 - \frac{\beta}{3!} p^3 + \ldots \tag{19}$$

and use of this to expand the left-hand side of eqn (12) into power series in p, we can determine the constants c_∞, υ and β by comparing the coefficients of same powers of p on both sides of eqn (12). After a lengthy algebra, one obtains

$$c_\infty^2 = g_\infty / \rho, \tag{20}$$

$$\upsilon = g_\infty \left(\frac{a_1 n_1}{g_{1\infty}} + \frac{a_2 n_2}{g_{2\infty}}\right) \tag{21}$$

$$\beta = (\omega n_1 n_2 c_\infty)^2 \left(\frac{\rho_1}{g_{2\infty}} - \frac{\rho_2}{g_{1\infty}}\right)^2 - \frac{3}{4} g_\infty \left[\frac{a_1 n_1}{g_{1\infty}}(3a_1 + 4T_1) + \frac{a_2 n_2}{g_{2\infty}}(3a_2 + 4T_2) + g_\infty \frac{n_1 n_2}{g_{1\infty} g_{2\infty}}(a_1 - a_2)^2\right] \tag{22}$$

where

$$\left.\begin{aligned} n_1 + n_2 = 1, \qquad & n_i = h_i/\omega, \\ \rho = \rho_1 n_1 + \rho_2 n_2, \qquad & \frac{1}{g_\infty} = \frac{n_1}{g_{1\infty}} + \frac{n_2}{g_{2\infty}} \end{aligned}\right\}. \tag{23}$$

We see that ρ and g_∞ are, respectively, the effective mass density and the effective equilibrium modulus of the composite.

When both layers are elastic, $a_i = 0$ and hence $\upsilon = 0$. Moreover, only the first term of β remains and $\beta \geq 0$. Notice that the first term of β is proportional to the difference in the impedances of the two layers and becomes zero when the difference in the impedances is zero. Since the dispersive nature of the composite comes from the impedance mismatch, the first term of β is responsible for the oscillatory nature of the stress response.

When one or both of the layers are viscoelastic, υ is positive and non-zero while β can be positive, negative or zero. Not only is υ responsible for the dissipative nature of the stress response, the second part of β is also responsible for the dissipation.

The case when both υ and β vanish will not be considered here.

4. ASYMPTOTIC SOLUTIONS

From eqns (19) and (14), we have

$$\sigma_1(x, t) = \frac{1}{2\pi i} \int_{Br} \frac{1}{p} \exp\left\{\left(t - \frac{x}{c_\infty}\right) p + \frac{x}{2c_\infty}\left(\upsilon p^2 + \tfrac{1}{3}\beta p^2 + \ldots\right) dp.\right. \tag{24}$$

We will assume that x is sufficiently large that the terms denoted by the dots can be ignored. We will also assume that $\beta \neq 0$. The case $\beta = 0$ will be discussed later. Let

$$\left.\begin{aligned} b &= \left(\frac{x|\beta|}{2c_\infty}\right)^{1/3}, \qquad \tau = \left(t - \frac{x}{c_\infty}\right)\Big/ b, \\ \gamma &= b\frac{v}{|\beta|} = \left(\frac{xv^3}{2c_\infty\beta^2}\right)^{1/3}. \end{aligned}\right\} \tag{25}$$

Equation (24) then takes the form

$$\sigma^{\pm}(\gamma, \tau) = \frac{1}{2\pi i}\int_{Br} \frac{1}{p}\, e^{\tau p + \gamma p^2 \pm \frac{1}{3}p^3}\, dp \tag{26}$$

where the subscript 1 of σ has been omitted and the $+$sign is for $\beta > 0$ and $-$sign for $\beta < 0$.

By taking the Bromwich contour L_1 as shown in Fig. 2, it is not difficult to show that

$$\sigma^{+}(\gamma, \tau) + \sigma^{-}(\gamma, -\tau) = 1. \tag{27}$$

We will therefore consider only the case $\beta > 0$ and hence the integral

$$\sigma(\gamma, \tau) = \frac{1}{2\pi i}\int_{Br} \frac{1}{p}\, e^{\tau p + \gamma p^2 + \frac{1}{3}p^3}\, dp. \tag{28}$$

Using the identity

$$\frac{1}{p}e^{\tau p} = \frac{1}{p} + \int_0^{\tau} e^{sp}\, ds \tag{29}$$

the integral in eqn (28) can be divided into two parts:

$$\sigma(\gamma, \tau) = I_1 + I_2, \tag{30}$$

$$I_1 = \frac{1}{2\pi i}\int_{Br} \frac{1}{p}\, e^{\gamma p^2 + \frac{1}{3}p^3}\, dp, \tag{31}$$

$$I_2 = \int_0^{\tau} ds \left\{\frac{1}{2\pi i}\int_{Br} e^{sp + \gamma p^2 + \frac{1}{3}p^3}\, dp\right\}. \tag{32}$$

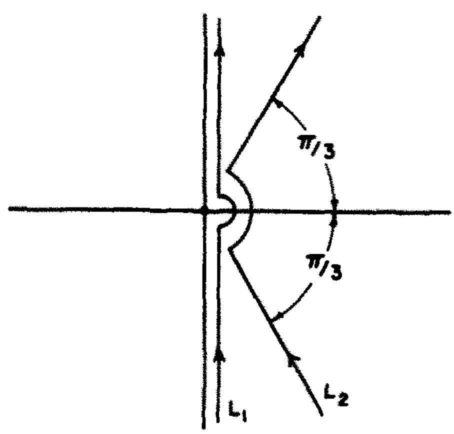

Fig. 2. Bromwich contours for eqns (28), (31) and (32).

Notice that $I_1 = \sigma(\gamma, 0)$ and hence is the magnitude of stress at $t = x/c_\infty$. By taking the Bromwich contour L_2, of Fig. 2, one obtains

$$I_1 = \frac{1}{3} + \frac{1}{\pi}\int_0^\infty \frac{1}{r} e^{-(1/2)\gamma r^2 - (1/3)r^3} \sin\left(\frac{\sqrt{3}}{2}\gamma r^2\right) dr \tag{33}$$

$$\cong \frac{1}{3} + \frac{\sqrt{3}}{2\pi}\left\{3^{-1/3}\Gamma\left(\frac{2}{3}\right)\gamma - 3^{1/3}\Gamma\left(\frac{4}{3}\right)\frac{\gamma^2}{2!} + \dots\right\}$$

where $\Gamma(x)$ is the Gamma function. If γ is very large, we take the Bromwich contour L_1 and obtain

$$I_1 = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{r} e^{-r^2} \sin\left(\frac{\delta}{3} r^3\right) dr$$

$$\cong \frac{1}{2} - \frac{\delta}{12\sqrt{\pi}}\left\{1 - \frac{35}{144}\delta^2 + \dots\right\} \tag{34}$$

where

$$\delta = \gamma^{-3/2} = \left(\frac{2c_\infty \beta^2}{x\upsilon^3}\right)^{1/2}. \tag{35}$$

We now turn to the integral I_2. By replacing the variable p by

$$p = z - \gamma, \tag{36}$$

eqn (32) can be written as

$$I_2 = \int_0^\tau e^{-s\gamma + \frac{2}{3}\gamma^3} ds \left\{\frac{1}{2\pi i}\int_{Br} e^{(s-\gamma^2)z + \frac{1}{3}z^3} dz\right\} \tag{37}$$

or

$$I_2 = \int_0^\tau e^{-s\gamma + \frac{2}{3}\gamma^3} Ai(-s + \gamma^2)\, ds \tag{38}$$

where the Airy function is defined as [8]

$$Ai(s) = \frac{1}{2\pi i}\int_{L_2} e^{-sz + \frac{1}{3}z^3} dz$$

$$= \frac{1}{\pi}\int_0^\infty \cos\left(sr + \tfrac{1}{3}r^3\right) dr. \tag{39}$$

Two extreme cases of $\gamma = 0$ and $\gamma = \infty$ have been studied in the literature. Before we evaluate σ for arbitrary γ, we will obtain these two extreme cases from eqn (30).

(1) $\gamma = 0$

For elastic composites, $\upsilon = 0$ and hence $\gamma = 0$. Equations (30), (33) and (38) then yield

$$\sigma(0, \tau) = \frac{1}{3} + \int_0^\tau Ai(-s)\, ds \tag{40}$$

This is precisely the asymptotic solution obtained in [1, 2]. The stress σ is an oscillatory function of τ (see Fig. 3).

(2) $\gamma = \infty$

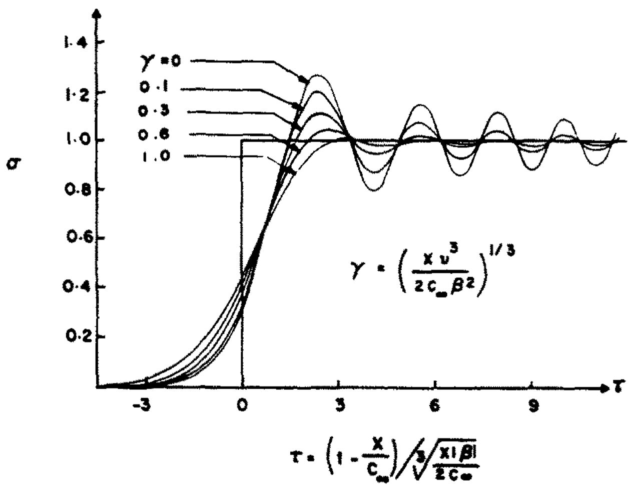

Fig. 3. Asymptotic solution for $0 \leq \gamma \leq 1$.

For viscoelastic composites, $\upsilon \neq 0$ and β of eqn (24) may or may not be zero. In [2] the term containing β was ignored. This is equivalent to assuming that $\beta = 0$ and hence $\gamma = \infty$. For a very large γ, the Airy function has the expression[8]:

$$Ai(s) \cong \frac{1}{2\sqrt{\pi}} \frac{1}{s^{1/4}} e^{-\frac{2}{3}s^{3/2}}. \tag{41}$$

Use of this expression in eqn (38) results in

$$I_2 \cong \frac{1}{2\sqrt{\pi}} \int_0^{\tau} e^{-s^2/(4\gamma)} \, d\left(\frac{s}{\sqrt{\gamma}}\right) = \frac{1}{2} \operatorname{erf}\left(\frac{\tau^*}{2}\right) \tag{42}$$

where

$$\left.\begin{aligned} \tau^* &= \left(t - \frac{x}{c_\infty}\right)\Big/ b^*, \qquad b^* = b\sqrt{\gamma} = \sqrt{\frac{x\upsilon}{2c_\infty}}, \\ \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} \, ds \end{aligned}\right\}. \tag{43}$$

Therefore when $\beta = 0$, (i.e. $\gamma = \infty$ or $\delta = 0$), eqns (34), (42) and (30) yield the following asymptotic solution obtained in [1, 2]:

$$\sigma = \tfrac{1}{2}\{1 + \operatorname{erf}(\tau^*/2)\} \tag{44}$$

The stress σ is a monotonically increasing function of τ^*, Fig. 4.

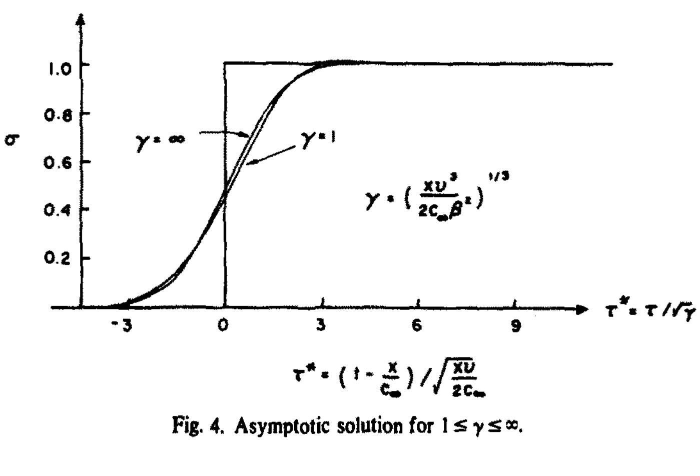

Fig. 4. Asymptotic solution for $1 \leq \gamma \leq \infty$.

5. NUMERICAL RESULTS AND DISCUSSION

For an arbitrary γ, the stress σ as a function of τ may be obtained from eqn (30) where I_1 is given by eqn (33) or (34) and I_2 is given by eqn (38). Since both I_1 and I_2 require a numerical integration, it might be simpler to evaluate σ directly from eqn (28). If we take L_2 of Fig. 2 as the Bromwich contour, eqn (28) reduces to

$$\sigma(\gamma, \tau) = \frac{1}{3} + \frac{1}{\pi} \int_0^\infty \frac{1}{r} e^{\frac{1}{2}(\tau r - \gamma r^2) - \frac{1}{3} r^3} \sin \left[\frac{\sqrt{3}}{2} (\tau r + \gamma r^2) \right] dr. \tag{45}$$

For the contour L_1, we have

$$\sigma(\gamma, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{1}{r} e^{-r^2} \sin \left(\tau^* r - \frac{\delta}{3} r^3 \right) dr \tag{46}$$

where τ^* and δ are defined in eqns (43) and (35). Notice that

$$\tau = \tau^* \quad \text{when} \quad \gamma = \delta = 1. \tag{47}$$

Notice also that because of the factor $(1/r)\,e^{-r^2}$ in the integrand of eqn (46), the absolute value of the integrand diminishes rapidly as r increases. For instance, at $r = 2$, $(1/r)\,e^{-r^2} \cong 9 \times 10^{-3}$ and at $r = 3$, $(1/r)\,e^{-r^2} \cong 4 \times 10^{-5}$. Therefore, the infinite integral can be replaced by an integral of finite interval say $0 \le r \le 3$. A similar argument applies to the integral in eqn (45).

Equation (45) is used to calculate σ for $\gamma = 0, 0.1, 0.3, 0.6$ and 1.0. The results are shown in Fig. 3. Equation (46) is used to calculate σ for $\gamma = 1$ and $\gamma = \infty$, (i.e. $\delta = 1$ and $\delta = 0$), Fig. 4. We see that the stress response differs very little for $\gamma = 1$ and $\gamma = \infty$.

The example of stress response at the 30th layer considered in [2] has a negative value of β and $\gamma = 0.58$. On the other hand, the example considered in ([3], p 100) has a positive value of β and $\gamma = 0.68$.

For a given viscoelastic composite, υ and β are known and fixed. γ then depends on x and increases as x increases. We see from Fig. 3 that the oscillatory nature of the stress diminishes as γ increases. Since for $\gamma \gtrsim 1$ the oscillation is practically non-existence, we may say that for

$$x \gtrsim \frac{2c_\infty \beta^2}{\upsilon^3} \tag{48}$$

the stress response is monotonic.

The asymptotic solution for viscoelastic composite derived in [3] is different from eqn (24). Using the notations of eqns (20)–(22), the asymptotic solution derived in [3] is based on the equation

$$\sigma(x, t) = \frac{1}{2\pi i} \int_{Br} \frac{1}{p} \exp \left\{ tp + \frac{\upsilon x}{2c_\infty} p^2 - \frac{xp}{c_\infty} \left(1 + \frac{\beta}{3} p^2 \right)^{-1/2} \right\} dp. \tag{49}$$

If we expand the last term in the exponent into a power series in p and ignore the terms of order higher than p^3, eqn (49) is identical to eqn (24). We are able to verify that υ in [3] is identical to the one obtained in eqn (21). However, β in [3] appears to be different from the expression in eqn (22).

CONCLUSION

The parameter γ defined in eqn (25) consists of the variables υ, β and x. The dissipative nature of the viscoelastic material is represented by υ and a part of β, while the remaining part of β represents the dispersive nature of the composite. The distance traveled by the wave is represented by x. Thus γ contains the influences on the wave profile due to dissipation, dispersion and the distance traveled by the wave. With γ determined from eqn (25), Figs. 3 and 4 provide the wave response.

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